

Calculation of the Moments and the Moment Generating Function for the Reciprocal Gamma Distribution

By Arne Fransén and Staffan Wrigge

Abstract. In this paper we consider the distribution $G(x) = F^{-1} \int_0^x (\Gamma(t))^{-1} dt$. The aim of the investigation is twofold: first, to find numerical values of characteristics such as moments, variance, skewness, kurtosis, etc.; second, to study analytically and numerically the moment generating function $\varphi(t) = \int_0^\infty e^{-tx} / \Gamma(x) dx$. Furthermore, we also make a generalization of the reciprocal gamma distribution, and study some of its properties.

Introduction. In [4] we considered, among many other things, the distribution $G(x) = F^{-1} \int_0^x (\Gamma(t))^{-1} dt$, which was suggested as a distribution of possible use in reliability theory by Dr. Gustaf Borenus, the former head of our section. He suggested this distribution because its shape looked like that of the gamma distribution, only with a thinner tail. Moreover, the problem as such was a mathematical challenge. The normalization constant F was calculated and presented to 60D by Fransén and Wrigge [4], to 80D by Fransén [3], and to 300D by W. A. Johnson [10].

We were asked by Dr. Samuel Kotz if we could calculate analytically and numerically the moments of the reciprocal gamma distribution. To our surprise, we managed to find not only a useful recurrence relation for the moments $\alpha_n = E(X^n)$ but also several analytical expressions.

The aim of this investigation is twofold: first, to find numerical values of characteristics such as moments, variance, skewness, kurtosis, etc.; second, to study analytically and numerically the moment generating function

$$\varphi(t) = \int_0^\infty (e^{-tx} / \Gamma(x)) dx.$$

Furthermore, we also make an obvious generalization of the reciprocal gamma distribution and study some of its properties.

When starting this study we did not know much about the work of our predecessors on similar problems. We learned that Ramanujan [8] as well as Wyman and Wong [17] and Dorning et al. [1] had studied related problems from a purely analytical standpoint. In Erdélyi et al., Vol. 3 [2, pp. 217-224] several functions close to the one studied by us are mentioned. Paley and Wiener used these functions, when studying inversion formulae for the Laplace transform; see [13, p. 39]. We also learned a lot when reading a doctoral thesis from 1887 by A. Lindhagen [11] on the gamma function.

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1. A General Formula for the Mean Value $E(\psi(X))$, Where X Has a Generalized Reciprocal Gamma Distribution. We begin with a

Definition. By a generalized reciprocal gamma distribution we mean a distribution with a density function $f(x) = 1/g(x)$, where $g(x)$ satisfies $g(x + 1) = xg(x)$ and $x \in [0, \infty)$.

We may then formulate

THEOREM 1. *Let $1/g(x)$ be a generalized reciprocal gamma probability density function, and let $\psi(x)$ be a continuous function such that*

$$E(\psi(X)) = \int_0^\infty \frac{\psi(x)}{g(x)} dx$$

exists. Then

$$E(\psi(X)) = \int_0^1 \frac{uF(u)}{g(u + 1)} du,$$

where

$$\begin{aligned} F(u) &= \psi(u) + \sum_{k=1}^\infty \frac{\psi(u + k)}{\prod_{j=0}^{k-1} (u + j)} \\ &= \psi(u) + \sum_{j=0}^\infty \frac{(-1)^j}{j!(u + j)} \left(\sum_{k=0}^\infty \frac{\psi(u + 1 + j + k)}{k!} \right) \end{aligned}$$

and the series for $uF(u)$ is assumed to converge uniformly in $[0, 1]$. \square

Proof.

$$\begin{aligned} E(\psi(X)) &= \sum_{k=0}^\infty \int_k^{k+1} \frac{\psi(x)}{g(x)} dx = \sum_{k=0}^\infty \int_0^1 \frac{\psi(u + k)}{g(u + k)} du \\ &= \int_0^1 \frac{uF(u)}{g(u + 1)} du. \end{aligned}$$

The ‘‘partial fraction’’ is deduced in the ordinary way using the identity

$$\frac{1}{u(u + 1) \cdots (u + n - 1)} = \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{(-1)^m}{(n-1)!} \frac{1}{u + m}, \quad n = 1, 2, \dots \quad \square$$

Of special importance is the case when $\psi(x)$ is periodic with a period equal to one. We may, e.g., determine the constant c_{2n} such that $c_{2n} \sin^{2n}(\pi x)/\Gamma(x)$ becomes a probability density. (See Section 3b.)

It should be noted that one might as well define the generalized reciprocal gamma distribution demanding that the density function $f(x)$ satisfies $f(x + 1) = f(x)/x$.

We have also studied the more general case where the probability density function $f(x)$ satisfies the functional relation

$$(1.1) \quad f(x + 1) = r(x)f(x),$$

where $r(x)$ is a rational, or even more general, function. It is then easy to see how to generalize Theorem 1.

An example of such a function $f(x)$, which we have studied in some detail, is

$$(1.2) \quad f(x) = c \prod_{i=1}^r \Gamma(m_i x) / \prod_{j=1}^s \Gamma(n_j x),$$

where m , and n , are positive integers, chosen such that the integral $\int_0^\infty f(x) dx$ exists, and c is a normalization constant. A simple example is

$$(1.3) \quad f(x) = c\Gamma(x)/\Gamma(2x),$$

giving $r(x) = \frac{1}{2}(2x + 1)^{-1}$.

We plan to present our studies of the more general density functions in (1.1) and (1.2), and corresponding moments and moment generating functions, in a future paper.

2. Some Formulae for the Moments and the Factorial Moments.

a. *The General Case.* Let X be a random variable with the same distribution as in Theorem 1. We want to obtain a useful formula for the moments $\alpha_n = E(X^n)$.

Let us define the numbers λ_n as $\lambda_n = \int_0^1 (x^n/g(x)) dx$. We may then formulate

THEOREM 2. *Let $\alpha_n = \int_0^\infty (x^n/g(x)) dx$ and $\lambda_n = \int_0^1 (x^n/g(x)) dx$. Then the moments α_n satisfy the recurrence relation $\alpha_n = 1 + \lambda_n - \lambda_0 + \sum_{k=1}^n \binom{n}{k} \alpha_{k-1}$, with starting value $\alpha_0 = 1$. \square*

Proof.

$$\alpha_n = \int_0^1 \frac{x^n}{g(x)} dx + \int_1^\infty \frac{x^n}{g(x)} dx = \lambda_n + \int_0^\infty \frac{(x+1)^n}{xg(x)} dx.$$

A simple use of the binomial theorem and some manipulations easily yield Theorem 2. \square

We may describe the solution of the equation in Theorem 2 in

THEOREM 3. *The solution of the recurrence relation in Theorem 2 may be written in the following way*

$$\alpha_n = b_{n+1} + \sum_{k=0}^n d_k(n)\lambda_k; \quad d_0(0) = 0, \quad d_n(n) = 1, \quad b_1 = 1,$$

where b_n is the Bell number of order n and the integers $d_k(n)$ satisfy

$$d_m(n+1) = \sum_{j=m}^n \binom{n+1}{j+1} d_m(j) - \delta_{0m}, \quad d_m(j) = 0 \quad \text{for } j < m. \quad \square$$

Proof. The numbers b_n satisfy the equation

$$(2.1) \quad b_{n+1} = \sum_{k=0}^n \binom{n}{k} b_k, \quad b_0 = 1$$

(see, e.g., Lunnon, Pleasants and Stephens [12, p. 2]), which is also obtained from the recurrence relation in Theorem 2. Now, consider

$$(2.2) \quad \alpha_{n+1} = 1 + \lambda_{n+1} - \lambda_0 + \sum_{j=0}^n \binom{n+1}{j+1} \left\{ b_{j+1} + \sum_{k=0}^j d_k(j)\lambda_k \right\}.$$

Identifying coefficients of λ_m on both sides of (2.2), we get

$$(2.3) \quad d_m(n+1) = \sum_{j=m}^n \binom{n+1}{j+1} d_m(j) - \delta_{0m}. \quad \square$$

The obvious advantage of knowing Theorem 3 is that in order to calculate $E(X^n)$ we only have to know the numbers λ_n . If $1/g(x)$ may be expanded as $1/g(x) = \sum_{k=1}^{\infty} c_k x^k$, then

$$(2.4) \quad \lambda_n = \sum_{k=1}^{\infty} c_k \frac{1}{n+k+1}.$$

Putting $\psi(x) = (x-1)(x-2) \cdots (x-n)$ in Theorem 1, we may state the following theorem for the factorial moments, viz.

THEOREM 4. Put $\psi(x) = \prod_{i=1}^n (x-i)$. Then

$$E(\psi(X)) = \int_0^1 \frac{\sum_{k=0}^{n-1} \prod_{i=1}^{n-k} (x-i) + 1}{g(x)} dx + e \int_0^1 \frac{xP(x)}{g(x+1)} dx,$$

where

$$P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)};$$

$$eP(x) = \frac{1}{x} + \frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \cdots. \quad \square$$

(Concerning the function $P(x)$ see, e.g., Lindhagen [11, pp. 21–22].)

We note the special case $n = 1$, which yields

$$(2.5) \quad \alpha_1 = 1 + \lambda_1 + e \int_0^1 \frac{xP(x)}{g(x+1)} dx.$$

From Theorem 2 we obtain $\alpha_1 = 2 + \lambda_1 - \lambda_0$. We conclude that

$$(2.6) \quad e \int_0^1 \frac{xP(x)}{g(x+1)} dx = 1 - \lambda_0.$$

b. *The Special Case.* In this case we put

$$\alpha_n = \frac{1}{F} \int_0^{\infty} \frac{x^n}{\Gamma(x)} dx$$

and define γ_n as $\gamma_n = F\lambda_n$, i.e.,

$$(2.7) \quad \gamma_n = \int_0^1 \frac{x^n}{\Gamma(x)} dx.$$

The recurrence relation in Theorem 2 then becomes

$$(2.8) \quad \alpha_n = 1 + \frac{\gamma_n - \gamma_0}{F} + \sum_{k=1}^n \binom{n}{k} \alpha_{k-1}; \quad \alpha_0 = 1.$$

When considering the moment generating function $E(e^{-tX})$ the numbers $\beta_n = \alpha_n/n!$ are more interesting. Thus we get, from a numerical point of view, the more useful recurrence relation

$$(2.9) \quad \beta_n = \frac{1}{n!} \left(1 + \frac{\gamma_n - \gamma_0}{F} \right) + \sum_{k=1}^n \frac{1}{k(n-k)!} \beta_{k-1}, \quad \beta_0 = 1.$$

To obtain the moments α_n we first calculate β_n using (2.9) and after that we use the identity $\alpha_n = n!\beta_n$.

When calculating the numbers γ_n we use the expansion

$$\frac{1}{\Gamma(x)} = \sum_{k=0}^{\infty} a_{k+1}x^{k+1}.$$

The coefficients a_{k+1} converge very fast towards zero and are tabulated by Fransén and Wriggle [4] with addendum by Fransén [3], in both cases to 80D. From (2.4) we get the formula

$$(2.10) \quad \gamma_n = \sum_{k=0}^{\infty} \frac{a_{k+1}}{n+k+2}, \quad n = 0, 1, 2, \dots$$

The values of $\alpha_n, \beta_n,$ and γ_n are presented in Table I for $n = 0, 1, 2, \dots, 15$ to 30D.

We also deduced a formula for the coefficients α_n using the Euler-Maclaurin summation formula and applied it for checking purposes.

Therefore consider the function

$$\varphi(t) = FE(e^{-tX}) = \int_0^{\infty} \frac{e^{-tx}}{\Gamma(x)} dx.$$

The Euler-Maclaurin summation formula with step-length $h = 1$ yields (B_{2k} are the ordinary Bernoulli numbers)

$$(2.11) \quad \varphi(t) = F \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \alpha_j = e^{-t+e^{-t}} + \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} a_{2k-1-j}.$$

(Cf. Section 3a.)

A remaining problem is how to calculate the coefficients δ_j in the expansion

$$(2.12) \quad e^{-t+e^{-t}} = Q(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \delta_j.$$

Differentiating (2.12) with respect to t we get $\dot{Q}(t) = -(1 + e^{-t})Q(t)$, which after identification of coefficients yields the relation

$$(2.13) \quad \delta_{j+1} = - \left(\delta_j + (-1)^j \sum_{k=0}^j (-1)^k \binom{j}{k} \delta_k \right), \quad \delta_0 = e.$$

From (2.11) we get the relation

$$(2.14) \quad F\alpha_j = (-1)^j \delta_j + \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} a_{2k-1-j}, \quad j = 0, 1, 2, \dots$$

Formulae (2.13) and (2.14) were used to check the numerical values of α_j calculated by (2.9) for small values of j ($j = 0, 1, 2, \dots, 6$).

From Hardy's lectures on Ramanujan [8, p. 196] or from his collected works, Volume IV [9, p. 544], we learn that

$$(2.15) \quad \int_0^{\infty} \frac{s^x}{\Gamma(1+x)} dx = e^s - \int_0^{\infty} \frac{x^{-1}e^{-sx}}{\pi^2 + \ln^2 x} dx, \quad s > 0.$$

Differentiating (2.15) $n + 1$ times with respect to s and putting $s = 1$, we get a formula for the factorial moments

$$(2.16) \quad \int_0^{\infty} \frac{\prod_{i=1}^n (x-i)}{\Gamma(x)} dx = e + (-1)^n \int_0^{\infty} \frac{x^n e^{-x}}{\pi^2 + \ln^2 x} dx.$$

Putting $s = e^{-t}$ in (2.15), we get, after differentiation with respect to t , the formula

$$(2.17) \quad \varphi(t) = \int_0^\infty \frac{e^{-tx}}{\Gamma(x)} dx = e^{-t+e^{-t}} + \int_0^\infty \frac{e^{-t-xe^{-t}}}{\pi^2 + \ln^2 x} dx.$$

(See also Section 3.)

To obtain an analytical expression for the moments α_n in this case, we must first consider a certain generating function, viz.

$$(2.18) \quad e^{xe^{-t}} = \sum_{k=0}^\infty C_k(x) (-1)^k \frac{t^k}{k!}.$$

For the functions $C_k(x)$ we easily obtain the relations

$$(2.19) \quad C_k(x) = \sum_{n=0}^\infty \frac{n^k}{n!} x^n,$$

$$(2.20) \quad C_k(x) = x \dot{C}_{k-1}(x), \quad k \geq 1,$$

$$(2.21) \quad C_k(x) = e^x P_k(x),$$

where $P_k(x)$ is a polynomial of order k with positive integer coefficients,

$$(2.22) \quad P_k(x) = \sum_{j=0}^k S(k, j) x^j;$$

$S(k, j)$ denotes Stirling's number of the second kind (Riordan [14, p. 192]),

$$(2.23) \quad C_n(u+v) = \sum_{k=0}^n \binom{n}{k} C_k(u) C_{n-k}(v).$$

Differentiating (2.17) n times with respect to t and putting $t = 0$, we obtain

$$(2.24) \quad F\alpha_n = \sum_{k=0}^n \binom{n}{k} C_k(1) + \int_0^\infty \frac{\sum_{k=0}^n \binom{n}{k} C_k(-x)}{\pi^2 + \ln^2 x} dx.$$

The value of $C_k(1)$ is eb_k , where b_k is the Bell number of order k (see Riordan [14, p. 193]). We note also that the numbers δ_j defined by (2.12) are related to $C_k(1)$ in the following way

$$(2.25) \quad \delta_n = (-1)^n \sum_{k=0}^n \binom{n}{k} C_k(1) = (-1)^n eb_{n+1}.$$

Finally, comparing (2.14) and (2.24), we obtain

$$(2.26) \quad \sum_{k=[(n+1)/2]+1}^\infty \frac{B_{2k}}{2k} a_{2k-1-n} = \int_0^\infty \frac{\sum_{k=0}^n \binom{n}{k} C_k(-x)}{\pi^2 + \ln^2 x} dx.$$

Gautschi [5] considered polynomials orthogonal with respect to the reciprocal gamma distribution. Let the polynomials be $\{\pi_k(x)\}$, normalized so that the coefficient of x^k equals 1. Gautschi tabulated the coefficients $\hat{\alpha}_k$ and $\hat{\beta}_k$ ($k = 0(1)39$, 18D) of the three-term recursion formula

$$(2.27) \quad \pi_{k+1}(x) = (x - \hat{\alpha}_k) \pi_k(x) - \hat{\beta}_k \pi_{k-1}(x); \quad \pi_0(x) = 1, \quad \pi_{-1}(x) = 0.$$

These polynomials could be used either to calculate or to check the moments of the reciprocal gamma distribution. No discrepancies were found.

c. *Some Population Characteristics in the Special Case.* For the sake of completeness only we give the numerical values of the population characteristics mentioned above to 30D. The moments around zero, α_n , are given in Table I, and from these we get

$$\begin{aligned}
 \text{mean value} &= \alpha_1 = && 1.93456\ 70421\ 47884\ 72118\ 37147\ 04369, \\
 \text{variance} &= \sigma^2 = \mu_2 = && 1.09393\ 63340\ 68611\ 31547\ 97434\ 13155, \\
 \text{skewness coefficient} &= \nu_1 = \mu_3/\sigma^3 = && 0.81591\ 45786\ 35844\ 85691\ 78575\ 43121, \\
 \text{coefficient of kurtosis} &= \nu_2 = \mu_4/\sigma^4 - 3 = && 0.80230\ 15983\ 14383\ 56469\ 69336\ 12202
 \end{aligned}
 \tag{2.28}$$

and the moment-ratios

$$\begin{aligned}
 g_3 &= \sqrt{\nu_1} = && 0.90327\ 99004\ 93664\ 73043\ 08181\ 09121, \\
 g_4 &= \nu_2 + 3 = && 3.80230\ 15983\ 14383\ 56469\ 69336\ 12202.
 \end{aligned}
 \tag{2.29}$$

3. Further Analysis of the Function $\varphi(t)$.

a. *Some Analytical Expressions.* Consider the function $\varphi(t) = FE(e^{-tX}) = \int_0^\infty (e^{-tx}/\Gamma(x)) dx$. Ramanujan, as mentioned by Hardy in [8], proved the formula

$$\int_{-\xi}^\infty \frac{y^x}{\Gamma(1+x)} dx + \int_0^\infty x^{\xi-1} e^{-yx} \left\{ \cos(\pi\xi) - \frac{\sin(\pi\xi)}{\pi} \ln x \right\} \frac{dx}{\pi^2 + \ln^2 x} = e^y,
 \tag{3.1}$$

when $y > 0$ and $\xi \geq 0$, using an ingenious method. Putting $\xi = 1$, we get

$$\int_{-1}^\infty \frac{y^x}{\Gamma(1+x)} dx - \int_0^\infty \frac{e^{-yx}}{\pi^2 + \ln^2 x} dx = e^y,$$

which may be written

$$\int_0^\infty \frac{y^{x-1}}{\Gamma(x)} dx = e^y + \int_0^\infty \frac{e^{-yx}}{\pi^2 + \ln^2 x} dx.
 \tag{3.2}$$

We proved (3.2) from scratch showing that the Laplace transform of the L.H.S. equals the Laplace transform of the R.H.S. Thereby we used probability, as well as residue, calculus.

Setting $y = e^{-t}$ in (3.2) yields

$$\varphi(t) = \int_0^\infty \frac{e^{-tx}}{\Gamma(x)} dx = e^{-t+e^{-t}} + \int_0^\infty \frac{e^{-t-xe^{-t}}}{\pi^2 + \ln^2 x} dx.
 \tag{3.3}$$

We will return to some different forms of Eq. (3.3) in Section 3c.

Several interesting analytical expressions of $\varphi(t)$ may be obtained using the Euler-Maclaurin summation formula. Thus we calculate

$$\varphi_1(t) = \sum_{k=1}^\infty \frac{e^{-tk}}{\Gamma(k)} = e^{-t+e^{-t}}
 \tag{3.4}$$

and

$$\varphi_2(t) = \sum_{k=0}^\infty \frac{e^{-t(k+1/2)}}{\Gamma(k + \frac{1}{2})} = \frac{e^{-t/2}}{\sqrt{\pi}} + (2N(2^{1/2}e^{-t/2}) - 1)e^{-t+e^{-t}},
 \tag{3.5}$$

where $N(\cdot)$ denotes the standardized normal distribution function, i.e.

$$N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-x^2/2} dx.$$

A formal use of the summation formula yields the equations

$$(3.6) \quad \varphi(t) = e^{-t+e^{-t}} + \sum_{j=0}^{\infty} \frac{(-1)^j t^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} a_{2k-1-j}$$

and

$$(3.7) \quad \varphi(t) = \frac{1}{2}(\varphi_1(t) + \varphi_2(t)) + \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} \left(\frac{1}{2}\right)^{2k} a_{2k-1-j}.$$

(Cf. Section 2b.) The more complicated case with the step-length $h = \frac{1}{4}$ is treated in a short note (Wrigge [16]).

We may generalize these results in

THEOREM 5. *Let $f(x) = 1/g(x)$ be a generalized reciprocal gamma probability density function such that $f(0) = 0$, $f'(0)$ is finite and nonzero and suppose that $1/g(x) = \sum_{k=1}^{\infty} c_k x^k$. Define $\kappa(t) = E(e^{-tX})$. Then formally we have, if $f^{(n)}(\infty) = 0$ for $n = 0, 1, 2, \dots$,*

$$\kappa(t) = \kappa_1(t) + \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} c_{2k-1-j}$$

and

$$\kappa(t) = \frac{1}{2}(\kappa_1(t) + \kappa_2(t)) + \sum_{j=0}^{\infty} \frac{(-t)^j}{j!} \sum_{k=[(j+1)/2]+1}^{\infty} \frac{B_{2k}}{2k} \left(\frac{1}{2}\right)^{2k} c_{2k-1-j},$$

where

$$\kappa_1(t) = \frac{e^{-t+e^{-t}}}{g(1)} \quad \text{and} \quad \kappa_2(t) = \frac{e^{-t/2}}{g(\frac{1}{2})} + \frac{\sqrt{\pi}}{g(\frac{1}{2})} (2N(2^{1/2}e^{-t/2}) - 1)e^{-t+e^{-t}}. \quad \square$$

Using Theorem 1, we may establish the following result.

THEOREM 6. *Let $f(x) = 1/g(x)$ be a generalized reciprocal gamma probability density function. Define $\kappa(t) = E(e^{-tX})$. Then*

$$\kappa(t) = \int_0^1 \frac{e^{-tu} {}_1F_1(1; u; e^{-t})}{g(u)} du,$$

where ${}_1F_1(\alpha; \gamma; x)$ denotes the confluent hypergeometric function (see Sneddon [15, p. 35]). An alternative expression is

$$\kappa(t) = \int_0^1 \frac{e^{-tu}}{g(u)} du + e^{-t+e^{-t}} \int_0^1 \frac{e^{-tu}}{g(u+1)} du + e^{-t+e^{-t}} \int_0^1 \frac{e^{-tu}}{g(u)} G(u) du,$$

where

$$G(u) = \sum_{k=1}^{\infty} \frac{(-1)^k e^{-tk}}{k!(u+k)}. \quad \square$$

The function $\varphi(t)$ is related to the beta function in a natural way. Consider the well-known identity

$$(3.8) \quad \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

We divide both sides by $\Gamma(x)\Gamma(y)$, multiply by $e^{-s(x+y)}$ and integrate with respect to x and y between 0 and ∞ . Formally we get

$$(3.9) \quad \int_0^1 \frac{\varphi(s - \ln(t))\varphi(s - \ln(1-t))}{t(1-t)} dt = \int_0^\infty \int_0^\infty \frac{e^{-s(x+y)}}{\Gamma(x+y)} dx dy = -\dot{\varphi}(s).$$

The last part of (3.9) will be proved using Eq. (3.12).

We will prove that

$$(3.10) \quad \varphi(t) = \int_0^\infty \int_0^\infty \frac{e^{-t(x+y)}}{\Gamma(x+y+1)} dx dy.$$

Therefore consider the auxiliary function

$$(3.11) \quad G_t(s) = \iint_{x+y \leq s; x, y \geq 0} \frac{e^{-t(x+y)}}{\Gamma(x+y+1)} dx dy.$$

Using the mean value theorem of integral calculus, we get

$$G_t(s + \Delta s) - G_t(s) = \frac{e^{-t(s+\theta\Delta s)}}{\Gamma(s + \theta\Delta s + 1)} \iint_{s \leq x+y \leq s+\Delta s} dx dy; \quad \theta \in [0, 1].$$

This easily yields

$$\frac{d}{ds} G_t(s) = \frac{se^{-ts}}{\Gamma(s+1)} = \frac{e^{-ts}}{\Gamma(s)}.$$

We finally get

$$(3.12) \quad G_t(\infty) = \varphi(t) = \int_0^\infty \frac{e^{-tx}}{\Gamma(x)} dx = \int_0^\infty \int_0^\infty \frac{e^{-t(x+y)}}{\Gamma(x+y+1)} dx dy.$$

The last part of (3.9) may now be proved using similar methods or simply by differentiating (3.10) with respect to t .

b. *Some Interesting Inequalities.* We start with Weierstrass' formula for $1/\Gamma(x)$, i.e.

$$(3.13) \quad \frac{1}{\Gamma(x)} = e^{\gamma x} \prod_{n=1}^\infty \left(1 + \frac{x}{n}\right) e^{-x/n}.$$

However, since $(1 + x/n)e^{-x/n} \leq 1$, we get

$$(3.14) \quad \frac{1}{\Gamma(x)} \leq e^{-xv_k} \sum_{j=0}^k c_{j,k} x^{j+1}, \quad \text{where } v_k = \sum_{n=1}^k n^{-1} - \gamma,$$

and the numbers $c_{j,k}$ are related to the Stirling numbers of the first kind, viz., $c_{j,k} = (s(k+1, j+1)/k!)(-1)^{j+k}$. (See e.g. Riordan [14, p. 90].) This yields the estimate

$$(3.15) \quad \varphi(t) \leq \sum_{j=0}^k c_{j,k} \frac{(j+1)!}{(t+v_k)^{j+2}}.$$

In Table II we give the values of the upper limit for $k = 24$ and $t = 0.0(0.1)5.0$.

Another inequality for $\varphi(t)$ may be obtained using the multiplication theorem for the gamma function. We see that

$$(3.16) \quad \varphi(t) = n \int_0^\infty \frac{e^{-ntx}}{\Gamma(nx)} dx, \quad n = 1, 2, 3, \dots$$

But it is known that $\Gamma(nx) = (2\pi)^{(1-n)/2} n^{nx-1/2} \Gamma(x) \prod_{k=1}^{n-1} \Gamma(x + k/n)$. This yields

$$\varphi(t) = n^{3/2} \int_0^\infty \frac{e^{-x(nt+n \log n)}}{(2\pi)^{(1-n)/2} \Gamma(x) \prod_{k=1}^{n-1} \Gamma(x + k/n)} dx.$$

Putting $\Gamma(x_0) = \min_{x>0} \Gamma(x)$ and $c = \Gamma(x_0)/\sqrt{2\pi}$, we may write

$$(3.17) \quad \varphi(nt + n \log n) \geq \frac{c^{n-1}}{n^{3/2}} \varphi(t), \quad n = 1, 2, 3, \dots$$

Similar inequalities may be deduced for the more general transformation

$$g(a, t) = \int_0^\infty \frac{e^{-axf(tx)}}{\Gamma(x)} dx.$$

Before continuing the analysis of $\varphi(t)$ we note the obvious inequality (from (3.3)),

$$(3.18) \quad \varphi(t) \geq e^{-t+e^{-t}}.$$

We will now deduce a different lower limit for $\varphi(t)$ using a variant of Jensen's inequality, viz., $\exp(\int_u^{u+1} \ln(f(x)) dx) \leq \int_u^{u+1} f(x) dx$, where $f(x)$ is supposed to be continuous and positive.

We put

$$G(t, u) = \int_u^{u+1} \ln \frac{e^{-tx}}{\Gamma(x)} dx$$

and get

$$(3.19) \quad G(t, u) = -t(u + \frac{1}{2}) - u \ln u + u - \ln \sqrt{2\pi}.$$

But

$$e^{G(t, u)} \leq \int_u^{u+1} \frac{e^{-tx}}{\Gamma(x)} dx.$$

Summing over $u = 0, 1, 2, \dots$, we obtain

$$(3.20) \quad \varphi(t) \geq \frac{e^{-t/2}}{\sqrt{2\pi}} \left(1 + \sum_{k=1}^\infty \frac{e^{k(1-t)}}{k^k} \right).$$

The simpler inequality (3.18) is slightly better in the interval (0, 0.6), see Table III.

We may easily generalize the result in (3.20), viz.,

THEOREM 7. *Let $f(x) = 1/g(x)$ be a generalized reciprocal gamma probability density function. Define $\kappa(t) = E(e^{-tX})$. Then*

$$\kappa(t) \geq K e^{-t/2} \left(1 + \sum_{k=1}^\infty \frac{e^{k(1-t)}}{k^k} \right),$$

where $K = e^{-\int_0^1 \ln(g(x)) dx}$. \square

An interesting example, which includes (3.20) as a special case, is given by the density function

$$f_{2n}(x) = c_{2n} \frac{\sin^{2n}(\pi x)}{\Gamma(x)}$$

(c_{2n} is a normalization constant and $c_0 = 1/F$), $n = 0, 1, 2, \dots$. Define $\kappa_{2n}(t) = E(e^{-tX})$ in this case. Then

$$(3.21) \quad \kappa_{2n}(t) \geq c_{2n} \frac{e^{-t/2}}{2^{2n}\sqrt{2\pi}} \left(1 + \sum_{k=1}^{\infty} \frac{e^{k(1-t)}}{k^k} \right).$$

Proof. Simply note that $\int_0^1 \ln(\sin(\pi x)) dx = -\ln 2$. \square

c. Numerical Calculation of $\varphi(t)$. We set ourselves the task to compute $\varphi(t)$ to 6D for $t = 0.0(0.1)5.0$. In all we considered more than 10 different methods to compute $\varphi(t)$ to the required accuracy. The methods used could roughly be divided into Gaussian quadrature rules and series expansions.

For small values of t there are several useful expansions. We may, e.g., consider

$$\varphi(t) = \int_0^1 \frac{e^{-tx}}{\Gamma(x)} dx + e^{-t} \int_0^{\infty} \frac{e^{-tx}}{x\Gamma(x)} dx,$$

from which we get the expansion (cf. Section 2b)

$$(3.22) \quad \varphi(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \gamma_n + e^{-t}(F - \gamma_0) + Fe^{-t} \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n} \beta_{n-1}.$$

More rewarding from a computational point of view was to use the Euler-Maclaurin expansions with step-lengths $h = 1$ resp. $h = \frac{1}{2}$ (formulae (3.6) and (3.7)). These expansions may in this case be regarded as the Taylor series expansions in disguise. Thus, (3.6) gave $\varphi(t)$ correct to 6D in the interval $[0, 2.0]$. Note that the somewhat complicated coefficients which occur in (3.6) may be calculated from or checked against (2.14). Equation (3.7) gave $\varphi(t)$ correct to 6D in the entire interval $[0, 5.0]$. The coefficients in (3.7) were truncated with an upper limit equal to 30 and calculated using high-precision techniques. (Note that we only know the numerical values of a_1, a_2, \dots, a_{61} ; see [3] and [4].) The achieved numerical results of $\varphi(t)$ using the Euler-Maclaurin methods are presented in Table IV.

For larger values of t it is useful to use Watson's Lemma. Applied to $\varphi(t)$, the lemma yields

$$(3.23) \quad \varphi(t) \sim \sum_{n=2}^{\infty} \frac{a_{n-1}(n-1)!}{t^n}, \quad \text{where } \frac{1}{\Gamma(x)} = \sum_{n=2}^{\infty} a_{n-1}x^{n-1}.$$

Values of $\varphi(t)$ correct to 6D were obtained from $t = 3.8$.

Finally, we turn our attention to the Gaussian quadrature rules. To be able to use a Gauss-Laguerre quadrature rule we rewrite (3.3) in the following two ways, viz.

$$(3.24) \quad \varphi(t) = e^{-t+e^{-t}} + \int_0^{\infty} \frac{e^{-u}}{\pi^2 + (t + \ln(u))^2} du$$

and

$$(3.25) \quad \varphi(t) = e^{-t+e^{-t}} + \int_0^{\infty} \frac{e^{-z+e^{-z}}}{\pi^2 + (t - z)^2} dz + \int_0^{\infty} \frac{e^{-z}e^{2z-e^z}}{\pi^2 + (t + z)^2} dz.$$

We evaluated numerically the integrals occurring in (3.24) and (3.25) using Gauss-Laguerre quadrature rules. However, none of the formulae were sufficient to yield 6D even when using a 15-point formula. The best result was achieved using (3.25) and a 15-point formula. The maximum absolute error was then about 4×10^{-4} in the required interval.

We considered the Gauss-Christoffel quadrature rule with respect to the weight distribution $w(x) = 1/\Gamma(x)$ on $[0, \infty)$, which is also mentioned by Gautschi [5]. In Table V we give weights $p_i(n)$ and abscissae $x_i(n)$ together with the remainder term coefficients c_n for $n = 14, 15$. Thus we get

$$(3.26) \quad \int_0^\infty \frac{f(x)}{\Gamma(x)} dx = \sum_{i=1}^n p_i(n) f(x_i(n)) + c_n f^{(2n)}(\xi), \quad 0 < \xi < \infty.$$

When calculating the weights and abscissae we used the well-known methods presented by Golub and Welsch [6] and Gustafson [7].

Using (3.26) with $f(x) = e^{-tx}$, $n = 14$ and 15 , and comparing the results with each other (and with other results), we could determine $\varphi(t)$ to 6D in the required interval. The numerical result for $n = 15$ appears in Table VI.

Note that the data in Table VI are subject to a progressively increasing error (beginning with $\varphi(2.8)$) so that the final entry is correct only to 6D. On the other hand, comparing Table IV with Table I in [16], we see that Table IV is correct to 10D.

The discrepancies between Table IV and Table VI may be interpreted in the following way. The function $\varphi(t)$ does not behave like an exponential polynomial for large values of the argument t . When using (3.26) with $f(x) = e^{-tx}$, we in fact put

$$(3.27) \quad \varphi(t) \approx \sum_{i=1}^n p_i(n) e^{-tx_i(n)},$$

and it is obvious from a comparison of Table IV and Table VI that this approximation is good only for “small” values of t . Therefore we think it is valuable to give Table VI with 10 decimals even if only 6 are correct throughout.

4. Tables. In this section we present the tables previously mentioned, i.e.,

I. The values of

$$\alpha_n = \frac{1}{F} \int_0^\infty \frac{x^n}{\Gamma(x)} dx, \quad \beta_n = \frac{\alpha_n}{n!}, \quad \gamma_n = \int_0^1 \frac{x^n}{\Gamma(x)} dx,$$

for $n = 0, 1, 2, \dots, 15$ to 30D.

II. Tabulation of the upper limit

$$UL_k(t) = \sum_{j=0}^k c_{j,k} \frac{(j+1)!}{(t+v_k)^{j+2}}$$

of $\varphi(t)$ for $k = 24$ and $t = 0.0(0.1)5.0$ ((3.15)).

III. Tabulation of the lower limits

$$LL_1(t) = e^{-t+e^{-t}} \quad \text{and} \quad LL_2(t) = \frac{e^{-t/2}}{\sqrt{2\pi}} \left(1 + \sum_{k=1}^\infty \frac{e^{k(1-t)}}{k^k} \right)$$

of $\varphi(t)$ for $t = 0.0(0.1)5.0$ ((3.18) and (3.20)).

TABLE I
Values of α_n , β_n and γ_n to 30D for n up to 15

Alfa (0) =	1.00000	00000	00000	00000	00000	00000
Beta (0) =	1.00000	00000	00000	00000	00000	00000
Gamma(0) =	0.54123	57343	28670	53014	95373	28880
Alfa (1) =	1.93456	70421	47884	72118	37147	04369
Beta (1) =	1.93456	70421	47884	72118	37147	04369
Gamma(1) =	0.35751	50224	23594	90711	31309	21362
Alfa (2) =	4.83648	59746	33426	89473	63606	92321
Beta (2) =	2.41824	29873	16713	44736	81803	46161
Gamma(2) =	0.26584	66316	55185	73050	08966	15849
Alfa (3) =	14.52263	23631	36764	25202	21329	96340
Beta (3) =	2.42043	87271	89460	70867	03554	99390
Gamma(3) =	0.21116	90393	30388	04296	03094	51101
Alfa (4) =	50.34552	58868	24648	33058	26933	58920
Beta (4) =	2.09773	02452	84360	34710	76122	23288
Gamma(4) =	0.17495	47281	85158	64133	35562	67979
Alfa (5) =	196.52960	95489	46802	68500	46071	32870
Beta (5) =	1.63774	67462	41223	35570	83717	26107
Gamma(5) =	0.14924	76440	33000	57064	19815	32525
Alfa (6) =	849.04403	89161	01902	55653	61461	70944
Beta (6) =	1.17922	78318	27919	30910	63002	03015
Gamma(6) =	0.13007	56150	34949	37665	14166	11576
Alfa (7) =	4008.05067	83135	15751	45650	77329	55599
Beta (7) =	0.79524	81504	59030	90306	67674	07332
Gamma(7) =	0.11523	86555	19321	63140	50448	25187
Alfa (8) =	20473.02093	44326	72186	66432	03391	11034
Beta (8) =	0.50776	34160	32556	35383	59206	43331
Gamma(8) =	0.10342	17446	83890	73215	80751	09877
Alfa (9) =	1 12278.68745	79147	99966	36906	13390	83314
Beta (9) =	0.30940	99632	32789	90290	55584	80322
Gamma(9) =	0.09379	13992	27884	14684	90322	19711
Alfa (10) =	6 56942.77694	75960	16930	75176	45786	40082
Beta (10) =	0.18103	58181	62366	62724	06172	19075
Gamma(10) =	0.08579	41856	48139	30346	42449	43023
Alfa (11) =	40 79302.42315	67110	12970	46671	96004	30938
Beta (11) =	0.10219	51264	41916	96260	64831	52948
Gamma(11) =	0.07904	84426	07206	82303	34826	57878
Alfa (12) =	267 63361.94933	30895	88698	48695	73219	27461
Beta (12) =	0.05587	32203	59458	27652	49604	32193
Gamma(12) =	0.07328	25617	57236	85865	02003	52079
Alfa (13) =	1848 15036.57385	64721	49672	31971	02966	00716
Beta (13) =	0.02967	95277	40433	51070	06021	75556
Gamma(13) =	0.06829	80580	98503	72710	45091	76512
Alfa (14) =	13388 81303.23715	11770	64912	67976	41416	82370
Beta (14) =	0.01535	79668	14990	18848	70268	33118
Gamma(14) =	0.06394	65552	27707	72516	40691	75135
Alfa (15) =	1 01461 22183.57558	18409	17108	15898	20647	91856
Beta (15) =	0.00775	89057	58085	16714	83684	75292
Gamma(15) =	0.06011	49366	30457	01942	63331	98573

TABLE II
Upper limit of $\varphi(t)$ to 6D

t	UL ₂₄	t	UL ₂₄	t	UL ₂₄
0.0	3.104091	1.7	0.307953	3.4	0.092232
0.1	2.545533	1.8	0.281396	3.5	0.087315
0.2	2.111334	1.9	0.257935	3.6	0.082767
0.3	1.769449	2.0	0.237128	3.7	0.078553
0.4	1.497054	2.1	0.218605	3.8	0.074642
0.5	1.277650	2.2	0.202057	3.9	0.071006
0.6	1.099143	2.3	0.187224	4.0	0.067621
0.7	0.952553	2.4	0.173885	4.1	0.064465
0.8	0.831128	2.5	0.161855	4.2	0.061519
0.9	0.729740	2.6	0.150973	4.3	0.058764
1.0	0.644448	2.7	0.141103	4.4	0.056184
1.1	0.572196	2.8	0.132128	4.5	0.053766
1.2	0.510592	2.9	0.123946	4.6	0.051497
1.3	0.457746	3.0	0.116470	4.7	0.049364
1.4	0.412154	3.1	0.109623	4.8	0.047358
1.5	0.372611	3.2	0.103338	4.9	0.045468
1.6	0.338142	3.3	0.097559	5.0	0.043687

TABLE III
Lower limits of $\varphi(t)$ to 6D

t	LL ₁	LL ₂	t	LL ₁	LL ₂	t	LL ₁	LL ₂
0.0	2.718282	2.625200	1.7	0.219300	0.266516	3.4	0.034506	0.079644
0.1	2.236333	2.163212	1.8	0.195011	0.243837	3.5	0.031123	0.075135
0.2	1.856570	1.800680	1.9	0.173699	0.223793	3.6	0.028081	0.070935
0.3	1.553978	1.513054	2.0	0.154948	0.206001	3.7	0.025342	0.067016
0.4	1.310384	1.282521	2.1	0.138409	0.190140	3.8	0.022877	0.063353
0.5	1.112412	1.095987	2.2	0.123787	0.175944	3.9	0.020656	0.059926
0.6	0.950100	0.943715	2.3	0.110832	0.163189	4.0	0.018654	0.056713
0.7	0.815940	0.818377	2.4	0.099333	0.151685	4.1	0.016850	0.053698
0.8	0.704215	0.714406	2.5	0.089107	0.141273	4.2	0.015222	0.050865
0.9	0.610528	0.627526	2.6	0.080000	0.131817	4.3	0.013754	0.048200
1.0	0.531464	0.554424	2.7	0.071877	0.123202	4.4	0.012429	0.045692
1.1	0.464344	0.492513	2.8	0.064623	0.115328	4.5	0.011233	0.043328
1.2	0.407055	0.439753	2.9	0.058136	0.108112	4.6	0.010153	0.041098
1.3	0.357912	0.394525	3.0	0.052329	0.101479	4.7	0.009178	0.038993
1.4	0.315561	0.355534	3.1	0.047125	0.095367	4.8	0.008298	0.037005
1.5	0.278909	0.321739	3.2	0.042458	0.089721	4.9	0.007502	0.035127
1.6	0.247065	0.292296	3.3	0.038269	0.084494	5.0	0.006784	0.033350

TABLE IV
Values of $\varphi(t)$ to 10D by the Euler-Maclaurin formula

t	$\varphi(t)$	t	$\varphi(t)$	t	$\varphi(t)$
0.0	2.8077702420	1.7	0.3009335120	3.4	0.0914210330
0.1	2.3262370474	1.8	0.2753948016	3.5	0.0865816184
0.2	1.9467718218	1.9	0.2527806955	3.6	0.0821024199
0.3	1.6443584989	2.0	0.2326807977	3.7	0.0779495893
0.4	1.4008236962	2.1	0.2147517803	3.8	0.0740930688
0.5	1.2027934333	2.2	0.1987050126	3.9	0.0705061033
0.6	1.0403059617	2.3	0.1842967120	4.0	0.0671648226
0.7	0.9058566158	2.4	0.1713200562	4.1	0.0640478828
0.8	0.7937313323	2.5	0.1595988329	4.2	0.0611361584
0.9	0.6995357300	2.6	0.1489822986	4.3	0.0584124762
1.0	0.6198584141	2.7	0.1393409958	4.4	0.0558613851
1.1	0.5520275472	2.8	0.1305633342	4.5	0.0534689567
1.2	0.4939329844	2.9	0.1225527820	4.6	0.0512226118
1.3	0.4438950131	3.0	0.1152255491	4.7	0.0491109691
1.4	0.4005665647	3.1	0.1085086674	4.8	0.0471237134
1.5	0.3628597079	3.2	0.1023383935	4.9	0.0452514796
1.6	0.3298899227	3.3	0.0966588753	5.0	0.0434857514

TABLE V
Abscissae $x_i(n)$ and weights $p_i(n)$ to 25S for $n = 14$ and 15

n	i	x_i	p_i
14	1	1.622858744052096654480454E-01	4.666936308723003950950625E-02
	2	5.229208541663484816942969E-01	2.646109639584808299341717E-01
	3	1.051449279617592254069644E+00	6.215442051821438630110274E-01
	4	1.726854794707082013337275E+00	8.142350693171694265817131E-01
	5	2.537139970455747796982883E+00	6.414191022297807547724504E-01
	6	3.477139770547848558819249E+00	3.099598643455755089011735E-01
	7	4.547183105381307655844180E+00	9.148075324998564823913511E-02
	8	5.75280924818442223580066E+00	1.612928241047227855479475E-02
	9	7.105517354977053201482566E+00	1.630797462009361385772756E-03
	10	8.624916011709722666870543E+00	8.849849557915662269500680E-05
	11	1.034350430938327785568881E+01	2.317750836324959203635878E-06
	12	1.231779194476028260112930E+01	2.446556157452933003045058E-08
	13	1.465920463327447787732363E+01	7.366805939154725353566559E-11
	14	1.765918673901601163881919E+01	2.653882898025302661903184E-14
15	1	1.490070588640254308122436E-01	3.924675165725691451042447E-02
	2	4.819351400658091188356546E-01	2.260891662923499176678823E-01
	3	9.723272335870747628473832E-01	5.527923916882637016023382E-01
	4	1.601001667819464486376536E+00	7.759201771782941429763990E-01
	5	2.356286743334194675877424E+00	6.757927480583987655263235E-01
	6	3.232350782165203111136036E+00	3.736746571000929295131507E-01
	7	4.227983287203939526165386E+00	1.312282470710664383753163E-01
	8	5.346137300427159830530296E+00	2.885258016254480346810935E-02
	9	6.594175177757317293887338E+00	3.860321080491039705432555E-03
	10	7.984927181287134330907394E+00	3.002822090665240753851297E-04
	11	9.539032836474949476943656E+00	1.265843407897613382797982E-05
	12	1.128985951391608900789341E+01	2.589555120399712100012903E-07
	13	1.329475808401580688647499E+01	2.136113605313739301854631E-09
	14	1.566620236805261061598493E+01	4.988209679385246841957279E-12
	15	1.869771150641537821275855E+01	1.356702577187496841722868E-15

$c_{14} = 3.8031842E-16$ $c_{15} = 1.3723708E-17$

TABLE VI
Values of $\varphi(t)$ to 10D by Gaussian quadrature rule

t	$\varphi(t)$	t	$\varphi(t)$	t	$\varphi(t)$
0.0	2.8077702420	1.7	0.3009335120	3.4	0.0914210319
0.1	2.3262370474	1.8	0.2753948016	3.5	0.0865816168
0.2	1.9467718218	1.9	0.2527806955	3.6	0.0821024175
0.3	1.6443584989	2.0	0.2326807977	3.7	0.0779495857
0.4	1.4008236962	2.1	0.2147517803	3.8	0.0740930635
0.5	1.2027934333	2.2	0.1987050126	3.9	0.0705060958
0.6	1.0403059617	2.3	0.1842967120	4.0	0.0671648119
0.7	0.9058566158	2.4	0.1713200562	4.1	0.0640478678
0.8	0.7937313323	2.5	0.1595988329	4.2	0.0611361378
0.9	0.6995357300	2.6	0.1489822986	4.3	0.0584124481
1.0	0.6198584141	2.7	0.1393409958	4.4	0.0558613473
1.1	0.5520275472	2.8	0.1305633341	4.5	0.0534689064
1.2	0.4939329844	2.9	0.1225527819	4.6	0.0512225456
1.3	0.4438950131	3.0	0.1152255490	4.7	0.0491108830
1.4	0.4005665647	3.1	0.1085086671	4.8	0.0471236023
1.5	0.3628597079	3.2	0.1023383931	4.9	0.0452513376
1.6	0.3298899227	3.3	0.0966588746	5.0	0.0434855716

IV. $\varphi(t) = \int_0^\infty (e^{-tx}/\Gamma(x)) dx$ tabulated to 10D for $t = 0.0(0.1)5.0$ by means of the Euler-Maclaurin expansion in (3.7).

V. The values of the numbers $p_i(n)$ and $x_i(n)$ corresponding to the weight function $1/\Gamma(x)$ (cf. (3.26)), $n = 14$ and 15 .

VI. $\varphi(t)$ tabulated to 10D for $t = 0.0(0.1)5.0$ by means of Gaussian quadrature (cf. (3.26)).

Comparing the values of Tables IV and VI, we thus may rely on all of them up to 6D.

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National Defence Research Institute
Division I, Section 123
S-102 54 Stockholm, Sweden

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